THE REPRESENTATION OF A FLEMING-VIOT PROCESS BY AN SPDE

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1 Introduction

In population genetics, Fleming-Viot processes play a major role combining suitably, in a unique setting, the main evolutionary forces which change the allelic gene frequencies in a population such as, mutation, natural selection and random genetic drift (see, e.g., \cite{6, 3, 5}). If $S$ is the set of allelic genes for one locus of interest in a population, then the Fleming-Viot process models the evolution of gene frequencies in time, assuming values in the state space $\mathcal{M}_1(S)$, which is the space of probability measures over $S$.

Along the same lines as in \cite{9}, we get a representation for this class of process by deriving a stochastic partial differential equation (SPDE) for the density of the random measure of the process. The main hindrance here is to deal with some sample-path properties for the jump-type Fleming-Viot process, which is tantamount to show that the state-space of the process, under some conditions, is contained in the set of absolute continuous measures with respect to the Lebesgue measure. Following Roelly-Coppoletta \cite{12} and Konno & Shiga \cite{9}, we restrict our study to the one-dimensional type space case.

The framework of an SPDE representing a measure-valued stochastic process may be useful in giving some insights, mainly for applications, even if its formulation is more restrictive than the formulation of a martingale problem, for instance. With respect to population genetics, it is notorious how many data is being collected in recent years, data that require theoretical models which could analyse them \cite{14, 15}, and the most disseminate approach (in any applied area) is still by the means of differential equations.

An outline of the paper follows. In Section 2 we recall the general definition of the jump-type Fleming-Viot process. By the means of a random time change which causes a mass change on the process, we obtain another jump-type measure-valued process in Section 3 that will aid us in proving Theorem 5.1. The establishment of the absolute continuity of the Fleming-Viot measure with respect to the Lebesgue measure on the real line is done in Section 4, from which we are able to write

$$ Y(dx, t) = Y(x, t)dx. \quad (1) $$

In Section 5 we prove Theorem 5.1 that allow us to write the process as a SPDE with jumps, mainly we show the existence of a Gaussian white noise measure related to the continuous part of the martingale measure. We give an application to population genetics in Section 6.
2 Jump-type Fleming-Viot process

We start with our traditional setup. Let $S$ be a one-dimensional metric space. Let $C(S)$ be the Banach space of continuous functions with the norm of the supremum ($\beta \in C(S), ||\beta|| = \sup_{x \in S} |\beta(x)|$). The set of Borel subsets of $S$ will be denoted $\mathcal{B}(S)$, and the vector space of bounded measurable functions on $S$, $b\mathcal{B}(S)$. Let $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ be the space of probability measures and the space of finite Radon measures over $\mathcal{B}(S)$, respectively. For $\beta \in b\mathcal{B}(S)$ and $\mu \in \mathcal{M}_1(S)$ (or $\mathcal{M}_R(S)$) we denote $\langle \beta, \mu \rangle = \int_S \beta(x)\mu(dx)$. To $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ it is given the weak topology. Let $\mathcal{B}(\mathcal{M}_1)$ denote the set of Borel subsets of $\mathcal{M}_1(S)$. Define also $b\mathcal{B}(\mathcal{M}_1)$, the vector space of bounded measurable functions on $\mathcal{M}_1(S)$. And let $\eta = \frac{\nu}{(1,0)}$, for $\eta \in \mathcal{M}_R(S)$.

Let $\Omega := D([0, \infty], \mathcal{M}_1(S))$ be the set of càdlàg\(^1\) functions from $[0, \infty]$ to $\mathcal{M}_1(S)$ equipped with the Skorokhod topology and $Y_t : \Omega \to \mathcal{M}_1(S)$ be the canonical process, $Y_t(\omega) = \omega(t)$. Let the $\sigma$-algebra $\mathcal{F}$ be the set of Borel subsets $\mathcal{B}(\Omega)$ and the filtration $\{\mathcal{F}_t \}_t$ in $\Omega$ be defined by $\mathcal{F}_0 := \cap_{c>0} \mathcal{F}_0^{y+c}$, where $\mathcal{F}_0 := \{\{Y_u \} : 0 \leq u \leq t\}$, and $\mathcal{F}_\infty := \vee_{n \in \mathbb{N}} \mathcal{F}_n$. Let $\mu \in \mathcal{M}_1(S)$, $a \geq 0$ and $\nu(du)$ be a measure in $\mathbb{R}$ such that $\int_0^\infty (u \wedge u^2) \nu(du) < \infty$. Fix also $g : [0, \infty] \to [0, \infty]$ a càdlàg function for which exists $\tau \in [0, \infty]$ such that $g(t) > 0$ for $t \in [0, \tau]$ and $g(t) = 0$ for $t \in [\tau, \infty]$. We consider a linear operator $\mathcal{L} : D(\mathcal{L}) \to b\mathcal{B}(S)$ which is the generator of a Feller semigroup $P_t : C(S) \to C(S)$. The process $\mathcal{P}_\mu = \{\mu \in \mathcal{M}_1(S)\} \subset \mathcal{M}_1(\Omega)$ is a jump-type Fleming-Viot process if:

- (FV1) $\mathcal{P}_\mu[Y_0 = \mu] = 1, \mathcal{P}_\mu[Y_t = Y_{\tau^-}, t \geq \tau] = 1;$(FV2) for $\beta \in D(\mathcal{L})$,

$$\langle \beta, Y_t \rangle = \langle \beta, Y_0 \rangle + \int_0^t \langle \mathcal{L}\beta, Y_s \rangle \mathbf{I}_{[s<\tau]} ds + M^c_t(\beta) + \int_0^t \int_{\mathcal{M}_R(S)} \frac{1}{g(s^-)} \frac{\langle 1, \eta \rangle \langle \beta, \eta - Y_s \rangle}{\langle 1, \eta \rangle} \mathbf{I}_{[s<\tau]} \tilde{N}(ds, d\eta),$$

(2)

is a càdlàg $\mathcal{F}_\tau$-semimartingale, where $\{M^c_t(\beta)\}_{t \geq 0}$ is a continuous martingale with predictable quadratic variation given by

$$\langle M^c_t(\beta) \rangle_t = a \int_0^t g(s)^{-1} \int_{x \in S} \beta(x) \delta_y Q(Y_s; dx \times dy) \mathbf{I}_{[s<\tau]} ds$$

(3)

with $Q(\mu; dx \times dy) = \delta_x(dy)\mu(dx) - \mu(dy)\mu(dx)$ and $\tilde{N}$ is a discontinuous $\mathcal{F}_\tau$-martingale measure corresponding to a random point process $N$ on $\mathbb{R}_+ \times \mathcal{M}_R(S)$, such that $\tilde{N}(t, B) = N(t, B) - \tilde{N}(t, B)$, where $\tilde{N}$ is the compensator of $N$ given by

$$\tilde{N}(ds, d\eta) = \left\{ \int_S \left[ \int_0^\infty \delta_{uw_s}(d\eta)\mu(du) \right] Y_s(dx) \right\} ds.$$  

(4)

**Theorem 2.1 (Hiraba, 2000)** Let $\mu \in \mathcal{M}(S)$. There exists a unique probability $\mathcal{P}_\mu$ on $\Omega$ satisfying the conditions FV1 and FV2 above.

\(^1\)Right-continuous with left limits.
Proof: See [7].

In population genetics, we interpret this process as follows. We consider $S$ the space of allelic genes for one locus. The frequencies of these genes in a population is modelled by $\mathcal{M}(S)$. Mutation is represented by the operator $L$, whose action may be seen as leading some kind of dynamic law over $S$. Random genetic drift is interpreted as composed of two terms: a continuous term given by $M^c$ which represents the loss in the population genetic diversity due to random mating; and a discontinuous term, led by $N$, due to abrupt changes suffered by the population. See [6, 2] for these interpretations and also [3, 5] for other factors that can be included in the model, such as, selection and gene conversion.

3 Random time change

The random time change performed here implies in a change of the mass of the process, thus originating a new measure-valued process which will be used to construct the jumping SPDE in Section 5.

Let $Z_t$ satisfy the following stochastic differential equation

$$dZ_t = \sqrt{\frac{a}{g(t)}} Z_t dB_t + c Z_t dt, \tag{5}$$

with $Z_0 = 1$, where $B_t$ is a Brownian motion independent of $Y_t$ (considered in an extension of $(\Omega, \mathcal{F}, \mathcal{P}_\mu)$, if necessary), and $c$ is a constant real number such that $c > \frac{a}{2g(t)}$, for $0 < t < \tau$. Thus $Z_t$ is equal to:

$$Z_t = \exp \left[ \int_0^t \sqrt{\frac{a}{g(s)}} dB_s + ct - \int_0^t \frac{a}{2g(s)} ds \right]. \tag{6}$$

Define now the random function $C_t : [0, \tau[ \rightarrow [0, \infty]$, by $C_t = \int_0^t Z_s ds$, that may be viewed as a random time change [4] or an occupation measure [8] weighted by $Z_s$.\footnote{Note that the stochastic differential equation is chosen in order that its solution $Z_t$ fits adequately in the proof of the Theorem 5.1.}

This random time change inspired by that of Shiga [13] became a technique for interchange between diffusions in $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ [9, 16].

An $\mathcal{M}_R$-valued process $X_t$ can be defined by

$$X_t(dx) = Z_{C_t^{-1}} Y_{C_t^{-1}}(dx). \tag{7}$$

We notice that $< 1, X_t > = Z_{C_t^{-1}}$ is positive and continuous in $t \geq 0$. A direct result from these definitions is the next lemma.

**Lemma 3.1** The random measure $Y_t(dx)$ is absolutely continuous with respect to the Lebesgue measure $\lambda(dx)$ for almost all $t > 0$ if, and only if, also is $X_t(dx)$.

\footnote{For each $A \in \mathcal{F}_t$, the weighted amount of time that $Z_t$ occupies $A$ during the interval $[0, t]$ is $C_t(A) = \int_0^t Z_s(A) ds.$}

3 Random time change
Define also $\tilde{F}_t \equiv F_{C_t^{-1}}$. From relation (7), we write $\langle \beta, X_t \rangle = Z_{C_t^{-1}} \langle \beta, Y_{C_t^{-1}} \rangle$, for $\beta \in D(L)$, to which we apply Itô’s formula to obtain

$$\langle \beta, X_t \rangle - \langle \beta, X_0 \rangle = \int_{[0,t]} \langle L\beta, X_s^- \rangle (1, X_s^-) ds + \tilde{M}_t(\beta) + c \int_{[0,t]} \langle \beta, X_s^- \rangle (1, X_s^-) ds.$$  

(8)

where

$$\tilde{M}_t(\beta) = \int_{[0,C_t^{-1}]} Z_s dM_s^\epsilon(\beta) + \int_{[0,C_t^{-1}]} \sqrt{\frac{a}{g(s^-)}} \langle \beta, Y_s^- \rangle Z_s dB_s$$

$$+ \int_{[0,C_t^{-1}]} \int_{\mathcal{M}_R(S)} Z_s \frac{\langle 1, \eta \rangle}{g(s^-) + \langle 1, \eta \rangle} \langle \beta, \eta - Y_s^- \rangle \tilde{N}(ds, d\eta)$$

(9)

is an $\tilde{F}_t$-martingale.

4 Absolute continuity w.r.t. Lebesgue measure

We admit the following conditions about the semigroup $P_t$:

**Conditions**:

(A1) $P_t$ admits a continuous density $p_t(x, y)$ in $(t, x, y) \in ]0, \infty[ \times S \times S$ with respect to the Lebesgue measure $\lambda(dx)$ on $S$,

(A2) there exists $0 < b < 1$ such that for every $T > 0$, $\sup_{0 \leq t \leq T} \sup_{x, y \in S} p_t(x, y)t^b < \infty$,

(A3) there exist constants $\delta > 0$ and $\epsilon > 0$ such that for every $T > 0$ we have $D_T > 0$ and for every $x, y \in S$ and $0 < h < 1$,

$$\int_0^t \int_S [p_{s+h}(z, x) - p_s(z, y)]^2 dz ds \leq D_T(|x - y|^\delta + h^\epsilon).$$

(10)

And now we may state the following result about the absolute continuity of the Fleming-Viot measure. The proof relies on the expressions for the **moments** of the process and will appear elsewhere.

**Theorem 4.1** Suppose that the semigroup $P_t$ satisfies the conditions (A1-A3). Then almost surely the random measure $Y_t(dx)$ of the one-dimensional jump-type Fleming-Viot process is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}$ for almost all $t > 0$.

5 Jumping SPDE

With the result of the absolute continuity of $Y_t$ in hand we may proceed to deduce a stochastic partial differential equation for the jump-type Fleming-Viot process. First, we recall the mathematical structure for an $S'(S)$-valued Wiener process following [9]. The space of **Schwartz distributions** $S'(S)$ is the topological dual of $S(S)$, the space of rapidly decreasing $C^\infty$-functions defined on $S$, equipped with the Schwartz topology. An $S'(S)$-valued continuous stochastic process $W_t$ defined
on a probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})\) is called an \(\mathcal{S}'(S)\)-valued standard \(\mathcal{F}_t\)-Wiener process, if for every \(\zeta \in \mathcal{S}(S)\), \(W_t(\zeta)\) is a one-dimensional \(\mathcal{F}_t\)-Brownian motion with the diffusion constant \(|\zeta||_{L_2(S)}\). The correspondent Gaussian random measure \(W(ds, dx)\) on \([0, \infty] \times S\) satisfies

\[
W_t(\zeta) = \int_0^t \int_S \zeta(x)W(ds, dx), \text{ for every } \zeta \in \mathcal{S}(S). \tag{11}
\]

**Theorem 5.1** Suppose that the semigroup \(P_t\) satisfies the conditions (A1-A3). There exists an \(\mathcal{S}'(S)\)-valued standard Wiener process \(W_t\) defined on an extension of the probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}_\mu)\) such that \(P_\mu\) almost surely

\[
Y_t(x) = Y_0(x) + \int_0^t \mathcal{L}Y_s(x)I_{[s<\tau]}ds + \int_0^t \sqrt{\frac{a}{g(s)}} \sqrt{Y_s(x)} \tilde{W}_s(x)I_{[s<\tau]}ds - \int_0^t \sqrt{\frac{a}{g(s)}} \int_{y \in S} Y_s(y) \tilde{W}_s(y)dyI_{[s<\tau]}ds + \int_0^t \int_{\mathcal{M}_t(S)} \frac{1}{g(s) + \langle 1, \eta \rangle} \left[ \frac{\partial \eta(x)}{\partial x} - Y_s^{-}(x) \right] I_{[s<\tau]}N(ds, d\eta). \tag{12}
\]

for every \(t \geq 0\) and \(\beta \in \mathcal{D}(\mathcal{L})\).

**Proof:** We make use of the processes \(Z_t, \ X_t\), the martingale measure \(\tilde{M}_t\) and their relation with \(Y_t\) presented in Section 3. Notice that by Theorem 4.1 and Lemma 3.1, the random measure \(X_t(dx)\) is absolutely continuous with respect to \(\lambda(dx)\). Let the martingale measure \(\tilde{M}^c(ds, dx)\) be related to \(\tilde{M}_t(\beta)\) via

\[
\int_0^t \int_S \beta(x) \tilde{M}^c(ds, dx) = \tilde{M}_t(\beta). \tag{13}
\]

Consider an \(\mathcal{S}'(S)\)-valued standard \(\mathcal{F}_t\)-Wiener process \(\tilde{W}_t\) independent of \(X_t\) and define also another an \(\mathcal{S}'(S)\)-valued process \(\hat{W}_t(\beta)\) by

\[
\tilde{W}_t(\beta) = \int_0^t \int_S \frac{1}{\sqrt{X_s(x)}} I_{x(s) \neq 0} \beta(x) \tilde{M}^c(ds, dx) + \int_0^t \int_S I_{x(s) = 0} \beta(x) \tilde{W}_s(ds, dx) \tag{14}
\]

defined thus in order to deal with \(X_t\) equal or not to zero. We may write this last relation as

\[
\int_0^t \int_S \sqrt{X_s(x)} \beta(x) \tilde{W}_s(x)dsdx = \int_0^t \int_S \beta(x) \tilde{M}^c(ds, dx) = \tilde{M}_t^c(\beta). \tag{15}
\]

We obtain from equation (8) and (15) the following equation for \(\langle \beta, X_t \rangle\)

\[
\langle \beta, X_t \rangle - \langle \beta, X_0 \rangle = \int_{[0,t]} \left[ \frac{1}{\langle 1, X_s^{-} \rangle} \mathcal{L}\beta \left( \frac{X_s^{-}}{\langle 1, X_s^{-} \rangle} \right) + c \frac{\langle \beta, X_s^{-} \rangle}{\langle 1, X_s^{-} \rangle} \right] ds + \int_0^t \int_S \sqrt{X_s(x)} \beta(x) \tilde{W}_t(x)dt dx + \int_{[0,C_t^{-1}]} \int_{\mathcal{M}_t(S)} Z_s \left( \frac{1}{g(s^{-})} + \langle 1, \eta \rangle \right) \langle \beta, \eta - Y_s^{-} \rangle N(ds, d\eta). \tag{16}
\]
Now, let a process \( W_t \) be given by
\[
\langle \beta, W_t \rangle = \int_0^t \frac{1}{\sqrt{\langle 1, X_s \rangle}} \sqrt{\frac{g(C_x^{-1})}{a}} \langle \beta, dW_s \rangle.
\] (17)

In such way, \( W_t \) is an \( \mathcal{S}'(S) \)-valued standard Wiener process and
\[
\sqrt{\frac{a}{g(t)}} d\langle \beta, W_t \rangle = \sqrt{Z_t} \langle \beta, d\hat{W}_t \rangle.
\] (18)

Applying Itô’s formula to \( \langle \beta, Y_t \rangle = \langle X_{C_1}, \beta \rangle Z_t^{-1} \) and using (18) we get
\[
\langle \beta, Y_t \rangle - \langle \beta, Y_0 \rangle = -\int_{[0,t]} \langle \beta, Y_s \rangle \left[ \frac{a}{g(s)} dB_s + c ds \right] + \int_{[0,t]} \sqrt{Y_s(x) \beta(x)} \sqrt{\frac{a}{g(s)}} W_s(x) ds dx
\]
\[
+ \int_{[0,t]} \int_{\mathcal{M}(S)} \langle 1, \eta \rangle \langle \beta, \eta - Y_s \rangle \tilde{N} ds, dx \end{align*}

Note then that as \( Z_t = \langle 1, X_{C_1} \rangle \), and using (16) with \( \beta(x) = 1 \), we have, by differentiating both members,
\[
dB_t = \int S \sqrt{\hat{Y}_t W_t} dt dx,
\] (20)

where we applied (18) again. Thus applying this last relation to (19) we can rewrite it as
\[
\langle \beta, Y_t \rangle - \langle \beta, Y_0 \rangle = -\int_{[0,t]} \frac{a}{g(s)} \langle \beta, Y_s \rangle \int_S \sqrt{Y_t W_t} dt dx + \int_{[0,t]} \frac{a}{g(s)} \int_S \sqrt{Y_s(x) \beta(x) W_s(x) ds dx}
\]
\[
+ \int_{[0,t]} \int_{\mathcal{M}(S)} \langle 1, \eta \rangle \langle \beta, \eta - Y_s \rangle \tilde{N} ds, dx \end{align*}

Bearing in mind that \( Y_t \) belongs to \( \mathcal{S}'(S) \) we get (12).

**q.e.d.**

**Remark 5.1** With respect to the equation (12), we note that it is a natural generalization of equation (0.6) of Konno & Shiga [9, p. 203], and we can obtain their equation by putting \( \nu = 0 \) and \( g(s) = a \) for all \( 0 \leq s \leq \tau \). This shows the consistency of our generalization.

**Remark 5.2** In the sense of Schwartz distributions, the density \( Y_t(x) \) can be viewed as satisfying the following equation
\[
\frac{\partial Y_t(x)}{\partial t} = \mathcal{L}^* Y_t(x) + \sqrt{\frac{a}{g(t)}} \int_{y \in S} \sqrt{Y_t(y)} \left[ \delta_x(y) - Y_t(x) \right] \dot{W}_t(y) dy
\]
\[
+ \int_{\mathcal{M}(S)} \langle 1, \eta \rangle \frac{\delta \eta(x)}{\delta x} - Y_t(x) \right] \tilde{N}_t(\eta) d\eta.
\] (22)

This equation differs from the equations commonly found in the literature [1, 11], since the jump part \( \tilde{N}(dt, d\eta) \) does not come from a Poisson point process.
6 An example from population genetics

Consider \( 0 \leq t_0 \leq t < \tau \). Now, let the mean density of type \( x \) at time \( t \) be given by

\[
m(t, x) = E[Y(t, x)].
\]  
\( \tag{23} \)

So, for \( \beta \in D(\mathcal{L}) \),

\[
E[\langle \beta, Y_t \rangle] = \int_{-\infty}^{+\infty} \beta(x)m(t, x)dx. \tag{24} \]

Using (22), we can easily derive the following equation in the weak form:

\[
\frac{\partial m}{\partial t} = \mathcal{L}^*m(t, x). \tag{25} \]

Let, then, \( m_2(t, x_1, x_2) \) be the mean joint density of \((x_1, x_2)\) at time \( t \), given by

\[
m_2(t, x_1, x_2) = E[Y(t, x_1)Y(t, x_2)]. \tag{26} \]

From the expression for the second moment of the process, we obtain

\[
\frac{\partial m_2}{\partial t} = \mathcal{L}^*_x m_2(t, x_1, x_2) + \mathcal{L}^*_x m_2(t, x_1, x_2) - 2\gamma^0_2(t)[m_2(t, x_1, x_2) - m(t, x_1)\delta_{x_1}(x_2)] \tag{27} \]

where \( \mathcal{L}^*_x \) is the adjoint operator of \( \mathcal{L}_x \) and \( \gamma^0_2(s) = a^2g(s) + 2\int_0^\infty \frac{u^2}{g(u) + u^2}\nu(du) \).

Denoting by \( \xi = x_2 - x_1 \) the difference between the types \( x_1 \) and \( x_2 \), we define the mean density of types differing by \( \xi \) as

\[
I(t, \xi) = \int_{-\infty}^{+\infty} m_2(t, x, x + \xi)dx \tag{28} \]

that satisfy the following equation

\[
\frac{\partial I}{\partial t} = \mathcal{L}^*_\xi I(t, \xi) - 2\gamma^0_2(t)[I(t, \xi) - \delta_0(\xi)]. \tag{29} \]

We restrict our attention to the Ohta-Kimura ladder model \([10]\) and put \( \mathcal{L} \equiv \frac{d^2}{dx^2} \). Fleming & Viot \([6]\) showed that, when there is no jump and there is homogeneity in time, for large times \( I(t, \xi) \) decays exponentially with \( \xi \) increasing. The compleat solution of (29), however, exhibit a variety of patterns, depending on the initial conditions. The solution \( I(t, \xi) \) may be written as

\[
I(t, \xi) = \int_{-\infty}^{+\infty} I(t_0, \xi - x)e^{-\xi^2/4(t-t_0)-2\gamma^0_2(t_0,t)}dx + \int_{t_0}^{t} \frac{e^{-\xi^2/4(t-s)-2\gamma^0_2(s,t)}}{\sqrt{2(t-t_0)}} \int_{-\infty}^{+\infty} 2\gamma^0_2(s)\sqrt{2\pi} ds. \tag{30} \]
References


